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GAUSSIAN FLUCTUATIONS FOR RANDOM WALKS IN RANDOM MIXING ENVIRONMENTS

BY

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ABSTRACT

We consider a class of ballistic, multidimensional random walks in random environments where the environment satisfies appropriate mixing conditions. Continuing our previous work [2] for the law of large numbers, we prove here that the fluctuations are Gaussian when the environment is Gibbsian satisfying the "strong mixing condition" of Dobrushin and Shlosman and the mixing rate is large enough to balance moments of some random times depending on the path. Under appropriate assumptions the annealed Central Limit Theorem (CLT) applies in both non-nestling and nestling cases, and trivially in the case of finite-dependent environments with "strong enough bias". Our proof makes use of the asymptotic regeneration scheme introduced in [2]. When the environment is only weakly mixing, we can only prove that if the fluctuations are diffusive then they are necessarily Gaussian.

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1. Introduction and main statements

1.1 INTRODUCTION. Fix an integer $d > 1$, let S denote the set of $2d$-dimensional probability vectors, and set $\Omega = S^{\mathbb{Z}^d}$. We consider all $\omega \in \Omega$ as an "environment" for the random walk that we define below in (1.1), and we denote by $\omega(z, \cdot) = \{\omega(z, z+e)\}_{e \in \mathbb{Z}^d, |e|=1}$ the coordinate of $\omega \in \Omega$ corresponding to $z \in \mathbb{Z}^d$. The random walk in the environment ω started at $z \in \mathbb{Z}^d$ is the Markov Chain $\{X_n\} = \{X_n; n \geq 0\}$ with state space $\mathbb{Z}^d$ such that $X_0 \equiv z$ and

$$(1.1) \qquad P_\omega^z(X_{n+1} = x + e | X_n = x) = \omega(x, x+e), \quad e \in \mathbb{Z}^d, \ |e| = 1.$$

Let P be a probability measure on Ω, stationary and ergodic with respect to the shifts in $\mathbb{Z}^d$. We denote by $\mathbb{P}^z = P \otimes P_\omega^z$ the joint law on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ of $\{X_n\}_n$ and ω. The process $\{X_n\}$ under $\mathbb{P}^z$ is called the random walk in random environment (RWRE). We will denote by $\mathbb{E}^z = E_{\mathbb{P}^z}$, $E_\omega^z = E_{P_\omega^z}$ the expectations corresponding to $\mathbb{P}^z, P_\omega^z$, respectively.

An important feature of random environments is the possible existence of traps, which are regions where the walk is drastically slowed down. (Traps do exist in the so-called nestling case, i.e., when condition $(\mathcal{A}4)$ below does not hold for any non-zero vector ℓ.) An essential difference between one and higher dimension is that the walk has to cross all the traps when $d = 1$, whereas it can go around them when $d > 1$.

Much is known about the RWRE when $d = 1$; see [16] for a recent review, including a discussion of laws of large numbers and central limit theorems for product and non-product measures P. Fluctuations have a normal limit or a stable limit law, according to the value of some parameter. See also [10] for recent stable limit results with $d = 1$ and non-product environments. In dimension $d > 1$, when P is a product measure and in the ballistic regime, i.e., when there exists a deterministic direction $\ell \in S^{d-1}$ such that $\limsup X_n \cdot \ell / n = v_\ell > 0$, the law of large numbers was first derived in the seminal paper [15] using a regenerative scheme. In the same context of P being a product measure, the central limit theorem for $\{X_n\}$ was obtained in [13], assuming uniform ellipticity and Kalikow's condition, using this regenerative scheme. Further development (in the ballistic case with P a product measure) can be found in [14].

In the case of dependent environment, laws of large numbers have been obtained in [7], [8], [12], with a rather mild dependence structure. (Also, a CLT is proved in [12] for certain environments exhibiting finite-range dependence.) More realistic dependence structures — including Gibbs measures in the mixing regime — were considered in [11] and [2], where the law of large numbers is proved. In [11], the author uses the approach of environment viewed from the point of view of the particle, while in our previous [2] we introduce a coupling method to find an asymptotic regenerative scheme.

Our goal in this paper is to adapt this latter technique to derive central limit theorems for the RWRE when P is not a product measure. This provides then, to our knowledge, the first example of RWRE's in dependent environments that do not exhibit finite range dependence, for which CLT type statements hold.

1.2 Some Assumptions: Mixing, Ellipticity, Drift. In the sequel, we fix an $\ell \in \mathbb{R}^d \setminus \{0\}$ such that ℓ has *integer coordinates*. With $\text{sign}(0) = 0$ and $\{e_i\}_{i=1}^d$ the canonical basis of $\mathbb{Z}^d$, let

$$\mathcal{E} = \{\text{sign}(\ell_i)e_i\}_{i=1}^d \setminus \{0\}. \tag{1.2}$$

Define the cone of vertex $x \in \mathbb{R}^d$, direction ℓ and angle $\cos^{-1}(\zeta), \zeta \in (0,1)$, by

$$C(x,\ell,\zeta) = \{y \in \mathbb{R}^d; (y-x)\cdot\ell \geq \zeta|y-x||\ell|\}. \tag{1.3}$$

We also need in the sequel the *truncated* ℓ cones defined as

$$C(x,\ell,\zeta,M) = \{y \in \mathbb{R}^d; y \in C(x,\ell,\zeta), (y-x)\cdot\ell \leq M\}, \tag{1.4}$$

as well as the notation, for $U \subset \mathbb{Z}^d$,

$$\mathcal{F}_U = \sigma((\omega_x)_{x\in U}). \tag{1.5}$$

We also let $|\cdot|$ denote the Euclidean distance in $\mathbb{R}^d$, and $\text{dist}(A,B)$ with $A, B \subset \mathbb{R}^d$ denotes the Euclidean distance between the sets A, B (we always consider $\mathbb{Z}^d$ as a subset of $\mathbb{R}^d$, so that the above is compatible with the ℓ_2 distance on $\mathbb{Z}^d$).

In [2], we made the following two assumptions on the environment:

Assumption 1.6:

($\mathcal{A}1$) *P is stationary and ergodic, and satisfies the following mixing condition on ℓ-cones: for all positive ζ small enough there exists a function $\phi(r)\to_{r\to\infty}0$ such that for any two events A, B with $P(A) > 0$, $A \in \sigma\{\omega_z; z\cdot\ell \leq 0\}$ and $B \in \sigma\{\omega_z; z \in C(r\ell,\ell,\zeta)\}$, it holds that*

$$\left|\frac{P(A\cap B)}{P(A)} - P(B)\right| \leq \phi(r|\ell|).$$

($\mathcal{A}2$) *P is elliptic and uniformly elliptic with respect to ℓ: $P(\omega(0,e) > 0; |e| = 1) = 1$, and there exists a $\kappa > 0$ such that*

$$P(\min_{e\in\mathcal{E}} \omega(0,e) \geq 2\kappa) = 1.$$

As described in [2], condition ($\mathcal{A}1$) is satisfied for a class of Gibbs random fields satisfying the so-called *weak mixing* condition of Dobrushin and Shlosman. For the strong CLT results, we will need a stronger notion of mixing, based on Dobrushin-Shlosman's *strong mixing* condition. We introduce next this notion, starting with the

Definition 1.7: Let $k \geq 1$, and let $\partial\Lambda^k = \{z \in \Lambda^c; \mathrm{dist}(z, \Lambda) \leq k\}$ be the k-boundary of $\Lambda \subset \mathbb{Z}^d$. A random field P is k-**Markov** if there exists a family π of transition kernels — called *specification* — $\pi_\Lambda = \pi_\Lambda(\prod_{y\in\Lambda} d\omega_y | \mathcal{F}_{\partial\Lambda^k})$ for finite $\Lambda \subset \mathbb{Z}^d$ such that

$$(1.8) \qquad P((\omega_x)_{x\in\Lambda} \in \cdot | \mathcal{F}_{\Lambda^c}) = \pi_\Lambda(\cdot | \mathcal{F}_{\partial\Lambda^k}), \quad P\text{-a.s.}$$

In addition, a k-Markov field P is called **strong-mixing** if there exist constants $\gamma > 0$, $C < \infty$ such that for all finite subsets $V \subset \Lambda \subset \mathbb{Z}^d$ with $\mathrm{dist}(V, \Lambda^c) > k$, and all $y \in \Lambda^c$,

$$(1.9) \qquad \begin{aligned} \sup\{\|\pi_\Lambda(\cdot \mid \omega) - \pi_\Lambda(\cdot \mid \omega')\|_V ; \omega, \omega' \in S^{\Lambda^c}, \omega_x = \omega'_x \ \forall x \neq y\} \\ \leq C \sum_{z\in\partial V^k} \exp(-\gamma|z-y|), \end{aligned}$$

with $\|\cdot\|_V = \|\cdot\|_{\mathrm{Var},V}$ the variational norm on V,

$$\|\mu - \nu\|_V = \sup\{\mu(A) - \nu(A); A \in \mathcal{F}_V\}.$$

The strong-mixing property holds for environments produced by a Gibbsian particle system at equilibrium in the uniqueness regime at high enough temperatures; see [3, 9]. Strong-mixing environments are weak-mixing, and by [2, Proposition 4.2], they are mixing on cones in the sense of Assumption ($\mathcal{A}1$). Summarizing this, we have ($\mathcal{A}1'$) $\Rightarrow$ ($\mathcal{A}1$), where we set

ASSUMPTION 1.10:
($\mathcal{A}1'$) *P is a Gibbs, strong-mixing, k-Markov field.*

We will also need some conditions on the environment ensuring the ballistic nature of the walk. Let U be a finite, connected subset of $\mathbb{Z}^d$, with $0 \in U$, and define on $U \cup \partial U$ an auxiliary Markov chain with transition probabilities

$$(1.11) \qquad \hat{P}_U(x, x+e) = \begin{cases} \dfrac{\mathbb{E}^o[\sum_{n=0}^{T_{U^c}} \mathbf{1}_{\{X_n=x\}} \omega(x, x+e) \big| \mathcal{F}_{U^c}]}{\mathbb{E}^o[\sum_{n=0}^{T_{U^c}} \mathbf{1}_{\{X_n=x\}} \big| \mathcal{F}_{U^c}]}, & x \in U, |e| = 1 \\ 1, & x \in \partial U, e = 0 \end{cases}$$

where $T_{U^c} = \min\{n \geq 0 : X_n \in \partial U^1\}$. This chain is known as Kalikow's Markov chain [6]. We will denote by $\hat{d}_U(x) = \sum_{|e|=1} e\hat{P}_U(x, x+e)$ the Kalikow drift, and by $d(x,\omega) = \sum_{|e|=1} e\omega(x, x+e)$ the RWRE's drift at x.

In addition to $(\mathcal{A}1)$ and $(\mathcal{A}2)$, we will assume one of the two following drift conditions, that ensure a ballistic behavior for the walk:

ASSUMPTION 1.12:

$(\mathcal{A}3)$ **Kalikow's condition**: *There exists a $\delta(\ell) > 0$ deterministic such that*

$$\inf_{U, x\in U} \hat{d}_U(x) \cdot \ell \geq \delta(\ell), \quad P\text{-a.s.}$$

(The infimum is taken over all connected finite subsets of $\mathbb{Z}^d$ containing 0.)

$(\mathcal{A}4)$ **Non-nestling condition**: *There exists a deterministic $\delta(\ell) > 0$ such that*

$$d(x,\omega) \cdot \ell \geq \delta(\ell), \quad P\text{-a.s.}$$

Clearly, $(\mathcal{A}4)$ is stronger than $(\mathcal{A}3)$. Both conditions, together with $(\mathcal{A}1, 2)$, imply that $\lim_{n\to\infty} X_n \cdot \ell = \infty$ $\mathbb{P}^o$-a.s.; see, e.g., [2, p. 887].

1.3 ASYMPTOTIC REGENERATIVE SCHEME. In this section, we recall some constructions and results from [2].

First, we define the RWRE on an enlarged space, depending on the vector ℓ with integer coordinates: instead of considering the law $\mathbb{P}^o = P \otimes P^o_\omega$ on the canonical space $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}^*}$, we consider the following probability measure,

$$\overline{\mathbb{P}}^o = P \otimes Q \otimes \overline{P}^o_{\omega,\varepsilon} \quad \text{on } \Omega \times W^{\mathbb{N}^*} \times (\mathbb{Z}^d)^{\mathbb{N}^*},$$

with $W = \{0\} \cup \mathcal{E}$ and $\mathcal{E}$ from (1.2): Q is a product measure, such that with $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ denoting an element of $W^{\mathbb{N}^*}$, $Q(\varepsilon_1 = e) = \kappa$, for $e \in \mathcal{E}$, while $Q(\varepsilon_1 = 0) = 1 - \kappa|\mathcal{E}|$. For each fixed ω, ε, $\overline{P}^o_{\omega,\varepsilon}$ is the law of the Markov chain $\{X_n\}$ with state space $\mathbb{Z}^d$, such that $X_0 = 0$ and, for every $z, e \in \mathbb{Z}^d$, $|e| = 1$,

(1.13)
$$\overline{P}^o_{\omega,\varepsilon}(X_{n+1} = z + e | X_n = z) = \mathbf{1}_{\{\varepsilon_{n+1}=e\}} + \frac{\mathbf{1}_{\{\varepsilon_{n+1}=0\}}}{1-\kappa|\mathcal{E}|}[\omega(z, z+e) - \kappa \mathbf{1}_{\{e\in\mathcal{E}\}}].$$

The point is that the law of $\{X_n\}$ under $Q \otimes \overline{P}^o_{\omega,\varepsilon}$ coincides with its law under P^o_ω, while its law under $\overline{\mathbb{P}}^o$ coincides with its law under $\mathbb{P}^o$.

We fix now, once and for all, a *particular* sequence $\bar{\varepsilon}$ with coordinates in $\mathcal{E}$, of length $|\ell|_1$, with sum equal to ℓ: for definiteness, we take $\bar{\varepsilon} = (\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_{|\ell|_1})$ with $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \cdots = \bar{\varepsilon}_{|\ell_1|} = \text{sign}(\ell_1)e_1, \bar{\varepsilon}_{|\ell_1|+1} = \bar{\varepsilon}_{|\ell_1|+2} = \cdots = \bar{\varepsilon}_{|\ell_1|+|\ell_2|} =$

$\mathrm{sign}(\ell_2)e_2, \ldots, \bar{\varepsilon}_{|\ell|_1-|\ell_d|+1} = \cdots = \bar{\varepsilon}_{|\ell|_1} = \mathrm{sign}(\ell_d)e_d$. We fix, through the whole paper, $\zeta > 0$ small enough such that

$$\bar{\varepsilon}_1, \bar{\varepsilon}_1 + \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_1 + \cdots + \bar{\varepsilon}_{|\ell|_1} = \ell \in C(0, \ell, \zeta), \tag{1.14}$$

and such that $(\mathcal{A}1)$ above is satisfied.

For $L \in |\ell|_1\mathbb{N}^*$ we will denote by $\bar{\varepsilon}^{(L)}$ the vector of length L consisting of the concatenation of $L/|\ell|_1$ copies of $\bar{\varepsilon}$, i.e.,

$$\bar{\varepsilon}^{(L)} = (\bar{\varepsilon}, \bar{\varepsilon}, \ldots, \bar{\varepsilon}).$$

Define

$$D' = \inf\{n \geq 0 : X_n \notin C(X_0, \ell, \zeta)\}. \tag{1.15}$$

Assumption $(\mathcal{A}3)$ (together with $(\mathcal{A}1, 2)$) implies that $\mathbb{P}^o(D' = \infty | \omega_x, x \cdot \ell \leq 0)$ is bounded away from 0; see [2, p. 887]. For all $L \in |\ell|_1\mathbb{N}^*$, set $\overline{S}_0 = 0$ and, using θ_n to denote time shift, set

$$\begin{aligned} \overline{S}_1 &= \inf\{n \geq L : X_{n-L} \cdot \ell > \max\{X_m \cdot \ell : m < n - L\}, \\ &\qquad (\varepsilon_{n-L}, \ldots, \varepsilon_{n-1}) = \bar{\varepsilon}^{(L)}\} \leq \infty, \\ \overline{R}_1 &= D' \circ \theta_{\overline{S}_1} + \overline{S}_1 \leq \infty. \end{aligned} \tag{1.16}$$

Note that the random times $\overline{S}_1, \overline{R}_1$, depend on both $\{X_n\}_n$ and $\{\varepsilon_n\}_n$. Define further, by induction for $k \geq 1$,

$$\begin{aligned} \overline{S}_{k+1} &= \inf\{n \geq \overline{R}_k : X_{n-L} \cdot \ell > \max\{X_m \cdot \ell : m < n - L\}, \\ &\qquad (\varepsilon_{n-L}, \ldots, \varepsilon_{n-1}) = \bar{\varepsilon}^{(L)}\} \leq \infty, \\ \overline{R}_{k+1} &= D' \circ \theta_{\overline{S}_{k+1}} + \overline{S}_{k+1} \leq \infty. \end{aligned}$$

These variables are stopping times for the pair $\{X_n, \varepsilon_n\}_n$ (depending on L), with

$$0 = \overline{S}_0 \leq \overline{S}_1 \leq \overline{R}_1 \leq \overline{S}_2 \leq \cdots \leq \infty,$$

and the inequalities are strict if the left member is finite. Also, since $X_n \cdot \ell \to \infty$ as $n \to \infty$, $\overline{S}_{k+1}$ is $\overline{\mathbb{P}}^o$-a.s. finite on the set $\{\overline{R}_k < \infty\}$. Define

$$\tau_1^{(L)} = \overline{S}_K \leq \infty, \quad \text{with } K = \inf\{k \geq 1 : \overline{S}_k < \infty, \overline{R}_k = \infty\} \leq \infty,$$

and

$$\tau_i^{(L)} = \tau_1^{(L)} \circ \theta_{\tau_{i-1}^{(L)}} + \tau_{i-1}^{(L)}, \quad i > 1.$$

The random time $\tau_1^{(L)}$ is the first time n when the walk performs as follow: at time $n-L$ it has reached a record value in the direction $+\ell$, then it travels using the ε-sequence only up to time n, and from time n on it does not exit the positive cone $C(X_n,\ell,\zeta)$ with vertex X_n. The advantage in considering $\tau_i^{(L)}$ is that at these times, the RWRE travels $|L|_1$ time units in the direction ℓ, **without learning any information about the environment**, allowing for decorrelation.

Under $(\mathcal{A}1,2,3)$, and if $\zeta \le \delta(\ell)/(3|\ell|)$, then $\tau_1^{(L)},\tau_2^{(L)},\ldots$ are finite $\overline{\mathbb{P}}^o$-a.s. for large L [2, Lemma 2.2 and p. 889]. For $L \in |\ell|_1\mathbb{N}^*$ we define $\tau_0^{(L)}=0$, and for $k \ge 1$,

$$\bar{\tau}_k^{(L)} = \kappa^L(\tau_k^{(L)} - \tau_{k-1}^{(L)}), \quad \overline{X}_k^{(L)} = \kappa^L(X_{\tau_k^{(L)}} - X_{\tau_{k-1}^{(L)}}). \tag{1.17}$$

(A rescaling by the factor κ^L is needed in order to keep the variables $\bar{\tau}_k^{(L)},\overline{X}_k^{(L)}$ of order 1 as $L\to\infty$.) The above random times yield an asymptotic (in the limit $L\to\infty$) regenerative structure, which can be expressed in term of the following coupling; see [2, Section 3]:

Coupling: We can enlarge once again our probability space [and we will continue to denote by $\overline{\mathbb{P}}^o$ annealed probabilities in this larger space], where the sequence $\{(\bar{\tau}_i^{(L)},\overline{X}_i^{(L)})\}_{i\ge1}$ is defined, in order to support also:

- a sequence $\{(\tilde{\tau}_i^{(L)},\tilde{X}_i^{(L)},\Delta_i^{(L)})\}_{i\ge1}$ of i.i.d. random vectors (with values in $\kappa^L\mathbb{N}^*\times\kappa^L\mathbb{Z}^d\times\{0,1\}$) where $\Delta_i^{(L)}\in\{0,1\}$ is such that

$$\overline{\mathbb{P}}^o(\Delta_i^{(L)}=1) = \phi'(L) := 2[\overline{\mathbb{P}}^o(D'=\infty)-\phi(L)]^{-1}\phi(L), \tag{1.18}$$

  such that the law of $(\tilde{\tau}_1^{(L)},\tilde{X}_1^{(L)})$ is identical to the law of $(\bar{\tau}_1^{(L)},\bar{X}_1^{(L)})$ under the measure $\overline{\mathbb{P}}^o[\cdot|D'=\infty]$,
- and another sequence $\{(Z_i^{(L)},Y_i^{(L)})\}_{i\ge1}$ such that

$$(\bar{\tau}_i^{(L)},\overline{X}_i^{(L)}) = (1-\Delta_i^{(L)})(\tilde{\tau}_i^{(L)},\tilde{X}_i^{(L)}) + \Delta_i^{(L)}(Z_i^{(L)},Y_i^{(L)}),$$

  and such that $\Delta_i^{(L)}$ is independent of $\{\tilde{\tau}_j^{(L)}\}_{j\le i-1},\{\tilde{X}_j^{(L)})\}_{j\le i-1}$, $\{\Delta_j^{(L)}\}_{j\le i-1}$ and of $(Z_i^{(L)},Y_i^{(L)})$.

The joint law of the variables $\{(Z_i^{(L)},Y_i^{(L)})\}_{i\ge1}$ is complicated, but $|Y_i^{(L)}|\le Z_i^{(L)}$ and $|\Delta_i^{(L)}Z_i^{(L)}|\le\bar{\tau}_i^{(L)}$. In [2], we used the following integrability condition (with $\alpha>1$):

ASSUMPTION 1.19:
($\mathcal{A}5^\alpha$) *There exists an* $M = M(L)$ *such that* $\phi'(L)^{1/\alpha'}M(L)^{1/\alpha}\longrightarrow_{L\to\infty}0$ *(with* $1/\alpha' = 1 - 1/\alpha$*), and*

$$\overline{\mathbb{P}}^o(\overline{\mathbb{E}}^o((\overline{\tau}_1^{(L)})^\alpha | D' = \infty, \mathcal{F}_0^L) > M) = 0, \tag{1.20}$$

where $\mathcal{F}_0^L = \sigma(\omega(y,\cdot) : \ell\cdot y < -L)$.

(Here and in the sequel, for a random variable X and measurable set A we use the notation $E(X|A,\mathcal{F}) = E(X;A|\mathcal{F})/P(A|\mathcal{F})$ whenever $P(A|\mathcal{F}) > 0$.) We postpone our comments on this assumption to below ($\mathcal{A}5'$) in the next section, and recall at this point the law of large numbers [2].

THEOREM 1: *Assume either* ($\mathcal{A}1,2,3$) *and* ($\mathcal{A}5^\alpha$) *for some* $\alpha > 1$, *or* ($\mathcal{A}1,2,4$). *Then, there exists a deterministic vector* v *with* $v\cdot\ell > 0$ *such that*

$$\lim_{n\to\infty}\frac{X_n}{n} = v, \quad \mathbb{P}^o\text{-a.s.}$$

1.4 MAIN RESULT. Under strong mixing assumptions of the form of Assumption ($\mathcal{A}1'$), we can give a full invariance principle for the RWRE, and a law of large numbers, under integrability conditions slightly weaker than ($\mathcal{A}5^\alpha$) with $\alpha > 2$. Namely, set

ASSUMPTION 1.21:
($\mathcal{A}5'$) *There exist an* $\alpha > 2$ *and* $M = M(L)$ *such that*

$$\overline{\mathbb{P}}^o(\overline{\mathbb{E}}^o((\overline{\tau}_1^{(L)})^\alpha | D' = \infty, \mathcal{F}_0^L) > M) = 0, \tag{1.22}$$

where $\mathcal{F}_0^L = \sigma(\omega(y,\cdot) : \ell\cdot y < -L)$.

We immediately note that, in the non-nestling case ($\mathcal{A}4$), conditions ($\mathcal{A}5^\alpha$) (for any $\alpha > 1$) and therefore ($\mathcal{A}5'$) always hold. Indeed, in that case, some exponential moments of $\overline{\tau}_1^{(L)}$ are bounded under the quenched measure $Q\otimes\overline{P}^o_{\omega,\varepsilon}$, uniformly in L and in the environment; see [2, (3.16)]. But our results go far beyond the non-nestling case.

Indeed, we give in [2, Section 5] various nestling examples where ($\mathcal{A}5^\alpha$) and ($\mathcal{A}5'$) hold: In the course of Theorem 5.1 therein, we prove that, under condition ($\mathcal{A}3$) with sufficiently large δ, $M(L)$ grows at most exponentially. More precisely, for $\delta > \delta_1(\kappa,\alpha)$,

$$M(L) \le e^{mL}, \quad L \ge 1, \tag{1.23}$$

with $m = m(\kappa,\alpha,\delta)$ finite. (The proof, given for $\alpha = 2$ in [2], extends to $\alpha > 2$.)

We can now state our main result:

THEOREM 2 (Annealed CLT, strong mixing): *Assume* $(\mathcal{A}1', 2, 3, 5')$. *Then there exist a deterministic, non-degenerate covariance matrix R and a deterministic vector v such that under $\mathbb{P}^o$, with $S_n(t) := [X_{[nt]} - vtn]/\sqrt{n}$, the path $S_n(t)$, taking values in the space of right continuous functions possessing left limits equipped with the supremum norm, converges weakly to a standard Brownian motion of covariance R.*

Note that when strong mixing is available, Theorem 2 yields the law of large numbers under weaker integrability assumptions than those used in Theorem 1.

Remark: It is worthwhile to note that the statement of Theorem 2 and its proof carry over to the case where P is the marginal on $S^{\mathbb{Z}^d}$ of a strong mixing Gibbs Markov field on $(S \times S')^{\mathbb{Z}^d}$ with S' any compact Polish space. For the sake of alleviating notation, we do not pursue this remark further.

Our results for mixing environments satisfying only $(\mathcal{A}1)$ are considerably weaker. With $M(L)$ from $(\mathcal{A}5^\alpha)$, $\phi'(L)$ from $(\mathcal{A}1)$ and (1.18), we will assume the existence of sequences $L = L(n)$ and $k_n = k(L(n), n) \longrightarrow_{n\to\infty} \infty$ such that

$$\frac{M(L)}{\kappa^{L\alpha/2} k_n^{(\alpha/2-1)}} \underset{n\to\infty}{\longrightarrow} 0, \tag{1.24}$$

and

$$M(L)^{1/\alpha} \phi'(L)^{1/\alpha'} \sqrt{\kappa^{-L} k_n} \underset{n\to\infty}{\longrightarrow} 0. \tag{1.25}$$

Finally, set $\tilde{T}_n^{(L)} = \sum_{i=1}^n \tilde{\tau}_i^{(L)}$.

THEOREM 3 (Annealed Gaussian behavior, weak mixing): *Assume* $(\mathcal{A}1, 2, 3, 5^\alpha)$ *with* $\alpha > 2$.

(a) *Assume further that sequences $L = L_n$ and k_n can be found that satisfy (1.24), (1.25), and the additional condition*

$$\frac{n}{\overline{\mathbb{E}}^o \tilde{T}_{k_n}^{(L)} \kappa^{-L}} \underset{n\to\infty}{\longrightarrow} 1. \tag{1.26}$$

Then, there exist a sequence of deterministic vectors $v(n)$, with $\lim_{n\to\infty} v(n) = v$, and a sequence of deterministic, positive definite, symmetric matrices R_n, defined in (3.20) below, such that with $R_n(w) = w^T R_n w$,

$$\lim_{n\to\infty} \left| \overline{\mathbb{P}}^o\Big(\frac{X_n \cdot w - nv(n)\cdot w}{\sqrt{n}} \le x\Big) - \overline{\mathbb{P}}^o\Big(\mathcal{N}(0, R_n(w)) \le x\Big) \right| = 0 \tag{1.27}$$

for all $x \in \mathbb{R}$ and all $w \in \mathbb{R}^d$. (According to the context, we denote by $\mathcal{N}(a, B)$ the Normal distribution of mean a and covariance matrix B, or a r.v. with this law.)

(b) *If (1.24) and (1.25) hold with $L \to \infty$, $k_n = k_L = e^{cL}$ for some constant $c > 0$, then one can find sequences $L_n \in |\ell|_1\mathbb{N}^*$ and $k_n = k(L_n, n)$ satisfying (1.24), (1.25) and (1.26).*

(c) *In the finite-dependence case (i.e., $\phi'(L) = 0$ for $L \geq L_0$), we can keep $L = L_0$ fixed. In this case, $v(n) = v$ and $R_n = R$, a positive definite matrix independent of n, and the statement is the standard central limit theorem:*

$$R^{-1/2}(X_n - nv)/\sqrt{n} \longrightarrow \mathcal{N}(0, \mathrm{Id}) \quad \textit{in law} .$$

Remark: In view of (1.23), we see that conditions $(\mathcal{A}1, 2, 3, 5^\alpha)$ with $\alpha > 2$, $\delta > \delta_1$ and $\phi(r) \leq e^{-\gamma r}$ with large enough γ ensure that part (b) of Theorem 3 applies. Hence, Theorem 3 applies to both non-nestling and nestling walks. On the other hand, we do not control in any way the convergence or non-degeneracy of the sequence of covariances R_n, and cannot rule out sub or super diffusive behavior in the generality of assumption $(\mathcal{A}1)$.

Before beginning the actual proof, let us give some guidance to the reader, and comparison with [15], [14] and [2].

The main idea in [15] is the introduction of regeneration times τ_i (corresponding, in the setup of the current paper, to $L = 0$ and $\zeta = 0$). When P is a product measure, the sequence of differences $(\tau_{i+1} - \tau_i, X_{\tau_{i+1}} - X_{\tau_i})$ is i.i.d., and the law of large numbers (in [15]) and CLT (in [14]), under $\mathbb{P}^o$, follow from the (non-trivial) analysis of moment bounds on $\tau_2 - \tau_1$. Unfortunately, when P is not a product measure, the sequence $\tau_{i+1} - \tau_i$ is not stationary anymore, let alone i.i.d., and a modification of the regeneration argument is needed.

In [2], we considered the case where P satisfies a mixing condition on cones of the form $(\mathcal{A}1)$. Using the representation in terms of the measure $\overline{\mathbb{P}}^o$, we introduced the regeneration times $\tau_i^{(L)}$ and the coupling with the i.i.d. sequence of times $\tilde{\tau}_i^{(L)}$. By taking first $n \to \infty$, followed by $L \to \infty$, and exploiting appropriate moment bounds, we were able to deduce a law of large numbers by showing that the errors $n^{-1}|\bar{\tau}_n^{(L)} - \tilde{T}_n^{(L)}|$, $n^{-1}|\bar{X}_n^{(L)} - \sum_{i=1}^n \tilde{X}_i^{(L)}|$ can be made arbitrarily small, when $n \to \infty$, by choosing L large.

Unfortunately, when trying to deduce a CLT in the mixing setup of [2], one has to control the quantities of interest at a much finer scale, viz. dividing by $\sqrt{n}$ instead of by n. In this case, merely comparing with an i.i.d. sequence is not good enough, even when L is large. In this paper, we propose two different strategies to deal with this problem.

In the first, leading to Theorem 2, we deal with strongly mixing Gibbs fields. We consider the past history of the path (and the "truncated cones" visited by the path between regeneration times), and describe the evolution of the path in the next truncated cone as a "chain with infinite connections", in the language of [5]. By using the strong mixing, we provide in Lemma 2.2 a control on the dependence of the evolution on the "distant past". This control is strong enough to allow us to compare the sequence of differences $(\tau_{i+1}^{(L)} - \tau_i^{(L)}, X_{\tau_{i+1}^{(L)}} - X_{\tau_i^{(L)}})$ with a stationary one determined by the (unique) invariant measure of the above mentioned chain. Standard results for such chains then allow us to deduce the CLT. In this approach, the parameter L is kept fixed.

The second strategy we describe, useful when the environment is weak but not strong mixing, is conceptually simpler: namely, we allow L to depend on n in such a way that it grows slowly enough so that error bounds valid for L fixed still apply, but fast enough such that the comparison with an i.i.d. sequence (of parameter $L = L_n$) is still good enough in the CLT scale. The main computation in this respect is contained in Lemma 3.4. This approach leads to Theorem 3, with the drawback that the mean and variance of the Gaussian approximation depend on n, leaving open the issue of having good bounds on the variance.

2. Proof of Theorem 2

The key to the proof in the strong mixing case is to consider the sequence of truncated cones of the environment produced by the regeneration times. To formalize this, define the space $\mathcal{T}$ of truncated cone environments and paths as

$$\mathcal{T} = \bigcup_{M = y\cdot\ell > 0, y \in \mathbb{Z}^d} \{M\} \times \mathcal{P}_M \times S^{C(0,\ell,\zeta,M)},$$

where the space of finite paths in the truncated cone $C(0,\ell,\zeta,M)$ (cf. (1.4)) is defined as

$$\mathcal{P}_M = \{\underline{x} = (x_1, \ldots, x_k) \in C(0,\ell,\zeta,M)^{\mathbb{N}^*} : x_0 = 0, |x_{i+1} - x_i| = 1\}.$$

Set $\overline{\mathcal{T}} = \mathcal{T} \cup \{\mathtt{s}\}$, where $\mathtt{s}$ is an extra stop symbol. We set $\mathcal{W} = \overline{\mathcal{T}}^{\mathbb{N}^*}$ as the space of infinite words consisting of finite truncated cone environments and finite cone based paths w, with the restriction that if $w_i = \mathtt{s}$ then $w_j = \mathtt{s}$ for all $j > i$. Note that finite words of length k can be naturally viewed as elements of $\mathcal{W}$ by setting $w_i = \mathbf{s}$ for all $i > k$. $\mathcal{W}$ inherits naturally a Borel structure that makes it into a measure space. We further define on $\mathcal{W}$ a lexicographic distance as

$$d(w,w') = 2^{-\min\{i: w_i \neq w'_i\}}.$$

Next, we fix L and note that the sequence $\tau_k^{(L)}$, $k \geq 1$, and the RWRE path X_n define an element $\underline{r} = (r_1, r_2, \dots) \in \mathcal{W}$ via

$$\begin{aligned} r_k = ((X_{\tau_{k+1}^{(L)}} - X_{\tau_k^{(L)}})\cdot\ell - \ell_L, \{X_{j+\tau_k^{(L)}} - X_{\tau_k^{(L)}}; j = 1, \dots, \tau_{k+1}^{(L)} - \tau_k^{(L)} - L\}, \\ \{\omega_y; y \in C(X_{\tau_k^{(L)}}, \ell, \zeta, (X_{\tau_{k+1}^{(L)}} - X_{\tau_k^{(L)}})\cdot\ell - \ell_L)\}), \end{aligned} \tag{2.1}$$

where $\ell_L = L|\ell|^2/|\ell|_1$ is an integer by our restriction on the allowed L and ℓ. We depict this construction in Figure 1. However, this will not be particularly useful to us as we think of $\mathcal{W}$ as a sequence of $\mathcal{T}$ valued symbols extending *backward* in time, and it will be convenient to think of $\underline{r}$ as defining a sequence of words $w^{(k)} = (r_k, r_{k-1}, \dots, r_1, \mathbf{s}, \mathbf{s}, \dots) \in \mathcal{T}^k$.

Further, recall from [2, pp. 889–890] the sigma-fields

$$\begin{aligned} \mathcal{H}_1 &= \sigma(\tau_1^{(L)}, X_0, \varepsilon_0, X_1, \dots, \varepsilon_{\tau_1^{(L)}-1}, X_{\tau_1^{(L)}}, \{\omega(y,\cdot); \ell\cdot y < \ell \cdot X_{\tau_1^{(L)}} - \ell_L\}), \\ \mathcal{H}_k &= \sigma(\tau_1^{(L)} \dots \tau_k^{(L)}, \quad X_0, \varepsilon_0, X_1, \dots, \varepsilon_{\tau_k^{(L)}-1}, X_{\tau_k^{(L)}}, \\ &\qquad \{\omega(y,\cdot); \ell\cdot y < \ell \cdot X_{\tau_k^{(L)}} - \ell_L\}), \end{aligned}$$

and set

$$\begin{aligned} \mathcal{U} = \{(m, y_1, \dots, y_m, \omega'); m \geq 1, y_i \in \mathbb{Z}^d, |y_{i+1} - y_i| = 1, y_m\cdot\ell > y_i\cdot\ell, \forall i < m, \\ \omega' \in S^{\mathbb{Z}^d \setminus \{z: z\cdot\ell > y_m\cdot\ell\}}\}. \end{aligned}$$

Then, $\overline{\mathbb{P}}^o$ induces a probability distribution $\mathbb{Q}^o$ on $\mathcal{U}$ such that, for $B \in \mathcal{H}_1$, $B = \bigcup_{t\in\mathbb{N}^*, z\in\mathbb{Z}^d} B_{t,z}$ with $B_{t,z} = B \cap \{\tau_1^{(L)} - L = t, X_{\tau_1^{(L)}-L} = z\}$, one has

$$\mathbb{Q}^o(B_{t,z}) = \overline{\mathbb{P}}^o((\tau_1^{(L)} - L, X_1, \dots, X_t, \{\omega_y\}_{y\cdot\ell\leq z\cdot\ell}) \in B_{t,z}),$$

and the law $\overline{\mathbb{P}}^o(\cdot|\mathcal{H}_1)$ induces on the sequence $\underline{r}$ a probability distribution such that the (random) kernels $h_{u,i}(\cdot|w_{i-1}, \dots, w_2, w_1)$, $u \in \mathcal{U}$ are well defined by the following: for each integer k, each measurable $A \subset \mathcal{T}^k$, and each measurable $B \in \mathcal{H}_1$,

$$\begin{aligned} &\overline{\mathbb{E}}^o[\mathbf{1}_B \overline{\mathbb{P}}^o((r_1, \dots, r_k) \in A|\mathcal{H}_1)] \\ &\qquad = \int_B \mathbb{Q}^o(du) \int_{\mathcal{T}} \cdots \int_{\mathcal{T}} \mathbf{1}_A \prod_{i=1}^k h_{u,i}(du_i|u_{i-1}, \dots, u_1). \end{aligned}$$

(To define the kernels $h_{u,i}$, simply note that that $\overline{\mathbb{P}}^o(r_k \in A|\mathcal{H}_k)$ defines a measurable function on $\mathcal{U} \times \mathcal{T}^{k-1}$, which is exactly $h_{u,i}(A|u_{i-1},\ldots,u_1)$.)

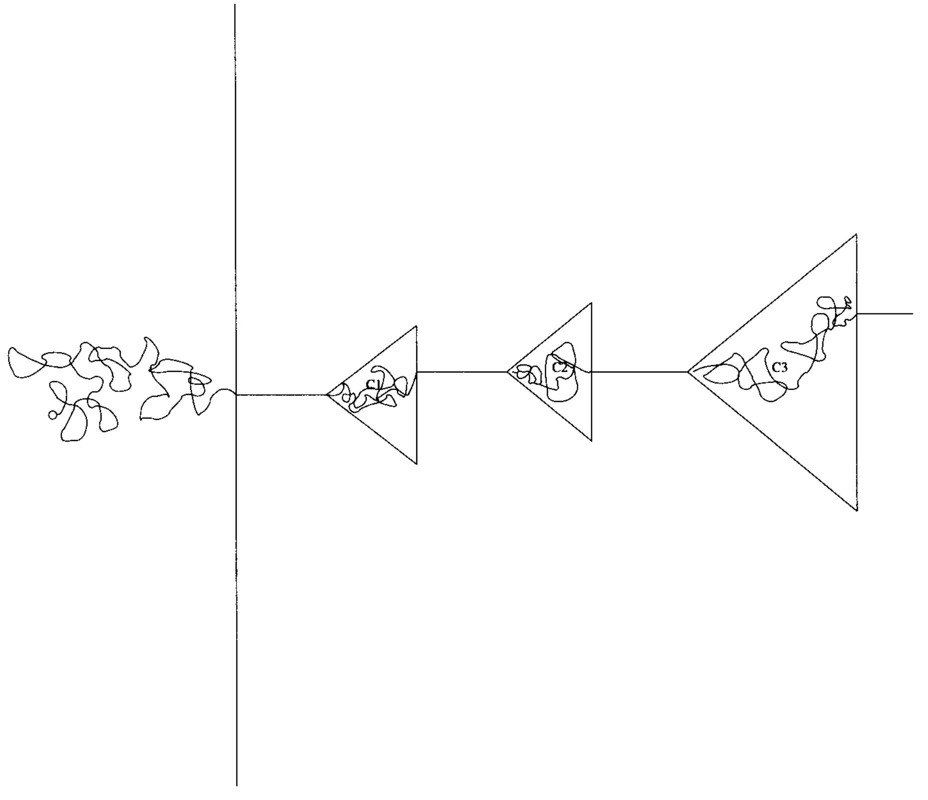

Figure 1. The sequence $\underline{r} = (r_1, r_2, \ldots)$, with $\ell = (1, 0, \ldots, 0)$. The hyperplane (to the right of the origin) is determined by the first regeneration location $X_{\tau_1^{(L)}}$, and $\mathcal{H}_1$ is determined by the path up to that location and the environment to the left of this first hyperplane. Shown are the cones C_1, C_2, C_3 as in the proof of Lemma 2.2, the random walk path inside the cones, and the directed paths between the cones (of length L) determined by the sequence ε.

The following lemma is crucial to our approach: Although the sequence $\underline{r}$ has complete dependence in the past, the influence of distant coordinates vanishes as rapidly as the correlations in the environment.

LEMMA 2.2: *Let* $i' \geq i$, $\underline{u}^{(i)} = (u_i, \ldots, u_1)$ *and* $\underline{u'}^{(i')} = (u'_{i'}, \ldots, u'_1)$ *be such* *that* $u_{i-j} = u'_{i'-j}$ *for* $j = 0, \dots i_0$. *Then,*

$$\sup_{u,u' \in \mathcal{U}} \|h_{u,i+1}(\cdot|\underline{u}^{(i)}) - h_{u',i'+1}(\cdot|\underline{u}'^{(i')})\|_{\mathrm{Var}} \le \phi(i_0 L). \tag{2.3}$$

Proof of Lemma 2.2: The proof is a modification of the argument in [2, Lemma 2.3], using the strong mixing assumption. Especially, the case $i = i' = i_0 = 1$ is a slight variation of the proof given in [2, Lemma 2.3].

For $\underline{u}^{(i)}, \underline{u}'^{(i')}$ from $\underline{u}, \underline{u}'$ infinite sequences in $\mathcal{T}$, we observe that the maximum over $i, i' \ge i_0$ of the left-hand side of (2.3) is achieved with $i = i' = i_0$, therefore we need to consider only the latter case.

We first note that the values $u, u_1, \dots, u_i$ determine a sequence of points $\bar{x}_i \in \mathbb{Z}^d$ and times $\bar{t}_i \in \mathbb{N}^*$ that encode the regeneration locations and times. More precisely, if $u = (m, y_1, \dots, y_m, \omega_{H_u})$ for the appropriate half space $H_u = \{x; x \cdot \ell \le y_m \cdot \ell\}$, and if $u_i = (m_i, x_1^{(i)}, \dots, x_{k_i}^{(i)}, \omega_{C_i})$ for some truncated cone C_i, we let p denote the projection on $\mathcal{T}$ given by $p(u_i) = (m_i, x_1^{(i)}, \dots, x_{k_i}^{(i)})$. Then, the regeneration locations and times are equal to

$$\bar{x}_0 = y_m + L\ell/|\ell|_1, \quad \bar{x}_i = \bar{x}_{i-1} + [x_{k_i}^{(i)} + L\ell/|\ell|_1], \quad \bar{t}_i = m + L + \sum_{j=1}^{i}[k_j + L].$$

In fact, from $(p(u), p(u_1), \dots, p(u_i))$, the whole path on the time interval $[0, \bar{t}_i]$ can be reconstructed: we denote by $\tilde{x} = \tilde{x}[(p(u), p(u_1), \dots, p(u_i))]$ this finite path — in particular, $\tilde{x}(\bar{t}_k) = \bar{x}_k$.

Let A be a measurable subset of $\mathcal{T}$, and write for short $\mathbf{1}_A = \mathbf{1}_{r_0 \in A}$, where r_0 is defined by (2.1) for $k = 0$, with $\tau_0^{(L)} = 0$.

Let also $F \ge 0$ be a $\mathcal{H}_1$-measurable bounded random variable [resp., $G \ge 0$ bounded measurable on $\sigma(r_1, \dots r_i)$]. Then for all $p_0 \in p(\mathcal{T}), \underline{p}^{(i)} \in p(\mathcal{T})^i$, there exist random variables F_{p_0} [resp., $G_{\underline{p}^{(i)}}$], measurable with respect to $\sigma(\{\omega(y, \cdot); y \cdot \ell < y_m \cdot \ell\}, \{\varepsilon_k; 1 \le k \le m\})$ [resp., $\sigma(\{\omega(y, \cdot); y \in U\}$] such that, on the event $\{p(r_0) = p_0\}$, $F = F_{p_0}$ [resp., on the event $\{p(r_k) = p_k, 1 \le k \le i\}$, $G = G_{\underline{p}^{(i)}}$]. Throughout, we use the notation $U = \bigcup_{j=1}^{i}(C_j + \bar{x}_{j-1})$, and we define the events $C(p_0) = \{X_k = \tilde{x}_k; 0 \le k \le \bar{t}_0\}$,

$$B(\underline{p}^{(i)}) = \{X_{k+\bar{t}_0} - X_{\bar{t}_0} = \tilde{x}_{k+\bar{t}_0} - \tilde{x}_{\bar{t}_0}; 0 \le k \le \bar{t}_i - \bar{t}_0\}$$

with $\tilde{x} = \tilde{x}[(p(u), p(u_1), \dots p(u_i))]$. By the Markov property,

$$\begin{aligned}
&\overline{\mathbb{E}}^o(FG\mathbf{1}_A \circ \theta_{\tau_{i+1}^{(L)}}) \\
&\quad = \overline{\mathbb{E}}^o(FG\mathbf{1}_{A \cap \{D' = \infty\}} \circ \theta_{\tau_{i+1}^{(L)}})
\end{aligned}$$

$$= \sum_{(p_0,\underline{p}^{(i)})\in p(\mathcal{T})^{i+1}} E_{P\otimes Q}\overline{E}^o_{\omega,\varepsilon}(F_{p_0}\mathbf{1}_{C(p_0)}G_{\underline{p}^{(i)}}\mathbf{1}_{B(\underline{p}^{(i)})}\mathbf{1}_{A\bigcap\{D'=\infty\}}\circ\theta_{\bar{t}_i})$$

$$= \sum_{p_0,\underline{p}^{(i)}} E_{P\otimes Q}\Big[\overline{E}^o_{\omega,\varepsilon}(F_{p_0}\mathbf{1}_{C(p_0)}G_{\underline{p}^{(i)}}\mathbf{1}_{B(\underline{p}^{(i)})})\times\overline{P}^{\bar{x}_i}_{\omega,\theta_{\bar{t}_i}\varepsilon}(A\bigcap\{D'=\infty\})\Big]$$

$$= \sum_{p_0,\underline{p}^{(i)}} E_{P\otimes Q}\Big[E_{P\otimes Q}(\overline{E}^o_{\omega,\varepsilon}(F_{p_0}\mathbf{1}_{C(p_0)}G_{\underline{p}^{(i)}}\mathbf{1}_{B(\underline{p}^{(i)})})$$

$$\times\overline{P}^{\bar{x}_i}_{\omega,\theta_{\bar{t}_i}\varepsilon}(A\bigcap\{D'=\infty\})\Big|\omega_z,z\in U)\Big]$$

$$= \sum_{p_0,\underline{p}^{(i)}} E_{P\otimes Q}\Big[\overline{G}_{p_0,\underline{p}^{(i)}}\times E_{P\otimes Q}(F_{p_0}\overline{P}^0_{\omega,\varepsilon}(C(p_0))$$

$$\times\overline{P}^{\bar{x}_i}_{\omega,\theta_{\bar{t}_i}\varepsilon}(A\bigcap\{D'=\infty\})\Big|\omega_z,z\in U)\Big],$$

where we have set

$$\overline{G}_{p_0,\underline{p}^{(i)}} = G_{\underline{p}^{(i)}}\overline{P}^0_{\omega,\varepsilon}(B(\underline{p}^{(i)})|X_l,l\le\bar{t}_0,X_{\bar{t}_0}=\bar{x}_0),$$

which is $\sigma(\omega_z, z\in U)$-measurable. Define $h_{\emptyset,i+1}(\cdot|\underline{u}^{(i)})$ the conditional law of r_{i+1} given $r_1,\dots r_i$, and define also ρ_A by

$$\rho_A = \sum_{p_0,\underline{p}^{(i)}} E_{P\otimes Q}\Big[\overline{G}_{p_0,\underline{p}^{(i)}}\times\mathrm{Cov}_{P(\cdot|\omega_z,z\in U)}\Big\{\overline{P}^{\bar{x}_i}_{\omega,\theta_{\bar{t}_i}\varepsilon}(A\bigcap\{D'=\infty\});$$
$$F_{p_0}\overline{P}^o_{\omega,\varepsilon}(C(p_0)\Big\}\Big] \tag{2.4}$$

allowing one to write

$$\overline{\mathbb{E}}^o(FG\mathbf{1}_A\circ\theta_{\tau^{(L)}_{i+1}}) = \rho_A + \sum_{p_0,\underline{p}^{(i)}} E_{P\otimes Q}[\overline{G}_{p_0,\underline{p}^{(i)}}$$
$$\times E_{P\otimes Q}\Big(\overline{P}^{\bar{x}_i}_{\omega,\theta_{\bar{t}_i}\varepsilon}\Big(A\bigcap\{D'=\infty\}\Big)\Big|\omega_z,z\in U\Big)$$
$$\times E_{P\otimes Q}(F_{p_0}\overline{P}^o_{\omega,\varepsilon}(C(p_0))|\omega_z,z\in U)] \tag{2.5}$$
$$= \rho_A + \sum_{p_0,\underline{p}^{(i)}} E_{P\otimes Q}[h_{\emptyset,i+1}(A|\underline{u}^{(i)})\overline{G}_{p_0,\underline{p}^{(i)}}E_{P\otimes Q}(F_{p_0}\overline{P}^o_{\omega,\varepsilon}(C(p_0))|\omega_z,z\in U)],$$

where we have used that

$$E_{P\otimes Q}\Big(\overline{P}^{\bar{x}_i}_{\omega,\theta_{\bar{t}_i}\varepsilon}\Big(A\bigcap\{D'=\infty\}\Big)\Big|\omega_z,z\in U\Big) = h_{\emptyset,i+1}(A|\underline{u}^{(i)})$$

holds on the set $B(\underline{p}^{(i)})\bigcap C(p_0)$, by definition of $h_{\emptyset,i+1}$ and since the sequence ε is i.i.d. Observe at this point that, by definition of $F_{p_0}, \overline{G}_{p_0}$,

$$
\begin{aligned}
&\overline{\mathbb{E}}^o(FGh_{\emptyset,i+1}(A|r^{(i)}))\\
&\quad=\sum_{p_0,\underline{p}^{(i)}}\overline{\mathbb{E}}^o(h_{\emptyset,i+1}(A|\underline{u}^{(i)})G_{\underline{p}^{(i)}}\mathbf{1}_{B(\underline{p}^{(i)})}F_{p_0}\mathbf{1}_{C(p_0)})\\
&\quad=\sum_{p_0,\underline{p}^{(i)}}\overline{\mathbb{E}}^o[h_{\emptyset,i+1}(A|\underline{u}^{(i)})\overline{G}_{p_0,\underline{p}^{(i)}}E_{P\otimes Q}(F_{p_0}\overline{P}^o_{\omega,\varepsilon}(C(p_0))|\omega_z, z\in U)].
\end{aligned}
\tag{2.6}
$$

Thus, (2.5) reads

$$\overline{\mathbb{E}}^o(FG\mathbf{1}_A\circ\theta_{\tau^{(L)}_{i+1}})=\rho_A+\overline{\mathbb{E}}^o(FGh_{\emptyset,i+1}(A|r^{(i)})). \tag{2.7}$$

The crucial point to observe is that for g measurable with respect to $\sigma(\omega_x, x\in C_{i+1}+\bar{x}_i)$, the strong mixing property (1.9) implies that, a.s.,

$$|E(g|\omega_x, x\in H_u\cup U)-E(g|\omega_x, x\in U)|\le\phi(iL)||g||_\infty,$$

with $\phi(r)=C'e^{-\gamma r/2}$. Hence, for f measurable with respect to $\sigma(\omega_x, x\in H_u)$, this results in

$$
\begin{aligned}
&|E(fg|\omega_x, x\in U))-E(f|\omega_x, x\in U)E(g|\omega_x, x\in U)|\\
&\qquad\qquad\le\phi(iL)E(|f||\omega_x, x\in U)||g||_\infty,
\end{aligned}
$$

replacing [2, (1.4)]. Hence, from (2.4) and (2.6)

$$|\rho_A|\le\phi(iL)\overline{\mathbb{E}}^o(FG).$$

Finally, one obtains from (2.7)

$$|\overline{\mathbb{E}}^o(FG\mathbf{1}_A\circ\theta_{\tau^{(L)}_{i+1}})-\overline{\mathbb{E}}^o(FGh_{\emptyset,i+1}(A|r^{(i)}))|\le\phi(iL)\overline{\mathbb{E}}^o(FG),$$

which is enough to prove the lemma. ∎

Lemma 2.2 allows us to show that the kernels $h_{u,k}(\cdot|\underline{w})$ converge as $k\to\infty$. With $M_1(\mathcal{T})$ denoting the space of probability measures on $\mathcal{T}$, we have

LEMMA 2.8: *There exists a measurable kernel h: $\mathcal{W}\mapsto M_1(\mathcal{T})$ such that*

$$\sup_{k\ge i, u\in\mathcal{U}, \underline{w}\in\mathcal{T}^{k-1}, w'\in\mathcal{W}: d(w,w')<2^{-i}}||h_{u,k}(\cdot|\underline{w})-h(\cdot|w')||_{\mathrm{Var}}<\phi(iL), \tag{2.9}$$

and

$$\sup_{w\in\mathcal{W}, w'\in\mathcal{W}: d(w,w')<2^{-k}}||h(\cdot|w)-h(\cdot|w')||_{\mathrm{Var}}<2\phi(kL). \tag{2.10}$$

Proof: Fix $u \in \mathcal{U}$ and $w = (w_1, w_2, \ldots) \in \mathcal{W}$, setting $w^{(k)} = (w_1, \ldots, w_k)$. Note that by Lemma 2.2, the sequence $(h_{u,k}(\cdot|w^{(k-1)}), k \geq 1)$ forms a Cauchy sequence with respect to the variation distance between elements of $M_1(\mathcal{T})$, with

$$\sup_{u,u',w\in\mathcal{W}} \|h_{u,k}(\cdot|w^{(k-1)}) - h_{u',k'}(\cdot|w^{(k'-1)})\|_{\mathrm{Var}} \leq \phi((k' \wedge k)L).$$

The existence of a limit $h_u(\cdot|w)$ follows from the completeness of $M_1(\mathcal{T})$, together with the estimate

$$\sup_{u,u',w\in\mathcal{W}} \|h_{u',k}(\cdot|w^{(k-1)}) - h_u(\cdot|w)\|_{\mathrm{Var}} \leq \sum_{k'\geq k} \phi(kL).$$

One deduces that h_u in fact does not depend on u, and the estimate in (2.10). ∎

We turn next to the construction of the "chain with infinite connections" with nice mixing properties alluded to above. Note that the kernel h and initial condition $w \in \mathcal{W}$ determine a Markov chain $\{w(n)\}_{n\geq 0}$ with state space $\mathcal{W}$, with law denoted $P_w(\cdot)$. Indeed, with $y \in \mathcal{T}$, $w \in \mathcal{W}$, define $yw \in \mathcal{W}$ by setting $(yw)_1 = y$ and $(yw)_i = w_{i-1}$ for $i \geq 2$. Then, with $w(n) \in \mathcal{W}$, let $y(n+1)$ be distributed according to $h(\cdot|w(n))$, and set $w(n+1) = (y(n+1)w(n))$.

By Lemma 2.8, the Markov chain thus defined satisfies conditions $\mathrm{FLS}(\mathcal{T}, 1)$ and M(1) of [5, pages 47, 51]. Hence, by [5, Theorem 2.27], it is uniformly ergodic and possesses a unique invariant distribution. Further, for $y \in \mathcal{T}$ with $y = (M, \underline{x}, \underline{s}) \in \mathcal{T}$ and $\underline{x} = (x_1, \ldots, x_m)$, define $f(y) = x_m$. Fatou's lemma and condition $(\mathcal{A}5')$ then imply the integrability condition

$$\sup_w \int |f(y)|^\alpha h(dy|w) < \infty. \tag{2.11}$$

Setting $g(y) = m$, the law of large numbers ([5, Proposition 4.1.1 and Theorem 4.1.2]) and another application of $(\mathcal{A}5')$ imply that

$$\frac{1}{n}\sum_{i=1}^{n} g(w(i)_1) \underset{n\to\infty}{\longrightarrow} C_1, \quad \frac{1}{n}\sum_{i=1}^{n} f(w(i)_1) \underset{n\to\infty}{\longrightarrow} C_2, \tag{2.12}$$

almost surely, with C_1, C_2 being deterministic and equal to the expectation of $g(w_1), f(w_1)$, respectively, under the unique invariant measure mentioned above.

Next, by [5, Theorem 4.1.5] and (2.11), and the ϕ mixing of the sequence $f(w(i)_1)$ ensured by [5, Theorem 2.1.5], the invariance principle holds, under

P_w, for $Z_n(t)$, where

$$Z_n(t) := \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} [f(w(i)_1) - C_2 g(w(i)_1)/C_1],$$

with variance that does not depend on the initial condition w.

It thus remains to transfer the statement of the invariance principle from the Markov chain $\{w(n)\}_{n\geq 0}$ to the original sequence S_n. Toward this end, define

$$\tilde{S}_n(k) = \frac{X_{\tau_k^{(L)}} - C_2 \tau_k^{(L)}/C_1}{\sqrt{n}},$$

and recall that by [2, Lemma 3.3], there exist deterministic positive sequences β_L and $\eta_L \to 0$ such that

$$\text{(2.13)} \qquad \limsup_{n\to\infty} n^{-1} |\tau_n^{(L)} - \beta_L \kappa^{-L} n| < \eta_L, \quad \overline{\mathbb{P}}^o\text{-a.s.}, \quad \liminf_{L\to\infty} \beta_L \geq t_{\mathrm{av}} > 0;$$

see [2, (3.5)] for the last fact. We assume throughout that L is chosen large enough such that both $\phi(L) < 1$ and $\eta_L < t_{\mathrm{av}}/2$.

Next, fix $\varepsilon \in (0,1)$ and $w \in \mathcal{W}$. Due to Lemma 2.8, and the fact that $\sum_k \phi(kL) < \infty$, one may find a sequence $k_0(\varepsilon) < \infty$ (with $k_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$) such that it is possible to construct a probability space with probability measure denoted $\tilde{\mathbb{P}}$ (both depending on ε, w) on which there exist:

- a sequence $(r_k)_k$ distributed according to $\overline{\mathbb{P}}^o(\underline{r} \in \cdot | \mathcal{H}_1)$, with $\underline{r}$ from (2.1),
- a sequence $w(n)$ distributed according to P_w,

such that

$$\text{(2.14)} \qquad \tilde{\mathbb{P}}(\exists k \geq k_0(\varepsilon) : r_k \neq w(k)_1) \leq \varepsilon.$$

Indeed, in view of Lemma 2.8, we can recursively couple $(r_k)_k$ and $(w(k))_k$ so that

$$\tilde{\mathbb{P}}(r_{i+1} = w(i+1)_1 | r_1, \ldots, r_i, w(1)_1, \ldots, w(i)_1) \geq 1 - \phi(kL)$$
$$\text{on } \{r_l = w(l)_1, i \geq l \geq i-k+1\}.$$

Then, (2.14) follows easily from $\sum_k \phi(kL) < \infty$.

Further, note that $\overline{\mathbb{P}}^o$-a.s., $\forall\delta > 0$,

$$\overline{\mathbb{P}}^o\Big(\sup_{k=1}^{n} \sup_{\tau_k^{(L)}\le t\le \tau_{k+1}^{(L)}} [|X_t - X_{\tau_k^{(L)}}|_1 + |t-\tau_k^{(L)}|] > 2\delta\sqrt{n}|\mathcal{H}_1\Big)$$

$$\begin{aligned}\text{(2.15)}\qquad &\le n\sup_{k=1}^{n}\overline{\mathbb{P}}^o(|\tau_{k+1}^{(L)}-\tau_k^{(L)}|\ge\delta\sqrt{n}|\mathcal{H}_k)\\ &\le n\operatorname{ess\,sup}\overline{\mathbb{P}}^o(\tau_1^{(L)}\ge\delta\sqrt{n}|D'=\infty,\mathcal{F}_0^L)\\ &\le\frac{n}{(\delta\sqrt{n})^\alpha}\operatorname{ess\,sup}\overline{\mathbb{E}}^o((\tau_1^{(L)})^\alpha|D'=\infty,\mathcal{F}_0^L)\\ &\le\frac{nM(L)\kappa^{-\alpha L}}{(\delta\sqrt{n})^\alpha}\underset{n\to\infty}{\longrightarrow}0.\end{aligned}$$

For any fixed T deterministic, set $J_T = 2(T+1)/t_{\mathrm{av}}\kappa^{-L}$. Note that, by construction and in view of (2.14),

$$\text{(2.16)}\quad \tilde{\mathbb{P}}\Big(\sup_{k_0(\varepsilon)\le k\le nJ_T}\Big|\tilde{S}_n(k)-\tilde{S}_n(k_0(\varepsilon))-Z_n\Big(\frac{k}{n}\Big)+Z_n\Big(\frac{k_0(\varepsilon)}{n}\Big)\Big|_1>0\Big)\le\varepsilon.$$

Further,

$$\text{(2.17)}\qquad \overline{\mathbb{P}}^o(\sup_{t\le\tau_1^{(L)}}|X_t|_1>\delta\sqrt{n})\le\overline{\mathbb{P}}^o(|\tau_1^{(L)}|>\delta\sqrt{n})\underset{n\to\infty}{\longrightarrow}0.$$

It follows from (2.15), (2.16) and (2.17), by taking first $n\to\infty$ and then $\varepsilon\to 0$, that the invariance principle for Z_n carries over to an invariance principle, under the measure $\overline{\mathbb{P}}^o$, for $\tilde{S}_n([tn])$, on the interval $0\le t\le J_T$, with the same nondegenerate limit covariance. On the other hand, by the law of large numbers (2.12) and (2.15),

(2.18)

$$\limsup_{n\to\infty}\overline{\mathbb{P}}^o\Big(\sup_{k\le nJ_T}\Big|\frac{\tau_k^{(L)}}{n}-C_2\frac{k}{n}\Big|>\delta\Big)$$
$$\le\limsup_{\varepsilon\to0}\limsup_{n\to\infty}\tilde{\mathbb{P}}\Big(\sup_{k\le nJ_T}\Big|\frac{\tau_k^{(L)}}{n}-C_2\frac{k}{n}\Big|>\delta\Big)=0,$$

while, by (2.14),

$$\text{(2.19)}\qquad \limsup_{n\to\infty}\overline{\mathbb{P}}^o(\tau_{nJ_T}^{(L)}<Tn)\le\limsup_{\varepsilon\to0}\limsup_{n\to\infty}\tilde{\mathbb{P}}(\tau_{nJ_T}^{(L)}<Tn)=0.$$

Hence, by the stability of the invariance CLT by random time changes [1, Theorem 14.4] together with (2.15), one concludes the invariance principle for $S_n(t)-C_2t/C_1$. ∎

3. Proof of Theorem 3

Throughout this section, we assume without further mention $(\mathcal{A}1, 2, 3, 5^\alpha)$ with $\alpha > 2$. We recall the coupling construction introduced above (1.18), and we stress that we take here $L = L(n) \to \infty$.

Fix a direction w as in the statement of part (a). The following preliminary lemma is easily proved.

LEMMA 3.1: *Assume (1.24). Then, with* $\tilde{T}_n^{(L)} = \sum_{i=1}^n \tilde{\tau}_i^{(L)}$,

$$\frac{\tilde{T}_{k_n}^{(L)}}{\overline{\mathbb{E}}^o \tilde{T}_{k_n}^{(L)}} \longrightarrow 1 \text{ in probability.} \tag{3.2}$$

Proof: Recall that

$$\overline{\mathbb{E}}^o \tilde{\tau}_1^{(L)} \geq t_{av}/2 > 0, \quad \text{for all } L \text{ large} \tag{3.3}$$

by [2, (3.5)]. Now $\tilde{T}_{k_n}^{(L)} = \sum_{j=1}^{k_n} \tilde{\tau}_j^{(L)}$, and the $\tilde{\tau}_j^{(L)}$ are i.i.d. by construction. Further, by $(\mathcal{A}5^\alpha)$,

$$\overline{\mathbb{E}}^o (\tilde{\tau}_1^{(L)})^2 \leq \overline{\mathbb{E}}^o ([\tilde{\tau}_1^{(L)^\alpha}])^{2/\alpha} \leq M(L)^{2/\alpha}$$

and hence, by (1.24),

$$\frac{\mathrm{Var}\tilde{T}_{k_n}^{(L)}}{(\overline{\mathbb{E}}^o \tilde{T}_{k_n}^{(L)})^2} \leq \frac{k_n M(L)^{2/\alpha}}{k_n^2 (t_{av}/2)^2} \underset{L\to\infty}{\longrightarrow} 0. \qquad \blacksquare$$

Set

$$\tilde{S}_m^{(L)} = \sum_{i=1}^m \tilde{X}_i^{(L)}, \quad v_L = \frac{\overline{\mathbb{E}}^o \tilde{X}_1^{(L)}}{\overline{\mathbb{E}}^o \tilde{\tau}_1^{(L)}},$$

and

$$r_L = r_L(w) = \mathrm{Var}(w \cdot [\tilde{X}_1^{(L)} - \tilde{\tau}_1^{(L)} v_L]).$$

We recall from [2, p. 895] that $v_L \to v$ as $L \to \infty$. For fixed $\delta > 0$, define

$$\tilde{Z}_\delta^{(L)}(w) = \tilde{Z}_\delta^{(L)}(w)(n) := \frac{w \cdot (\tilde{S}_{k_n}^{(L)} - \tilde{T}_{k_n}^{(L)} v_L)}{\sqrt{k_n}(\sqrt{r_L} \vee \delta \kappa^L)}.$$

The following lemma is the basic ingredient for the Gaussian approximation. Let us denote by $\mathcal{L}(Z)$ the law of a random variable Z, and by ρ the Prohorov distance between probability measures.

LEMMA 3.4: *Assume k_n satisfies (1.24), and set*

$$A_\delta(w) := \{n : r_L > \delta \kappa^L\}.$$

Then,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \rho\Big(\mathcal{L}(\tilde{Z}_\delta^{(L)}(w)), \mathcal{N}\Big(0, \Big(\mathbf{1}_{A_\delta(w)}(n) + \mathbf{1}_{A_\delta^c(w)}(n)\frac{r_L}{\delta\kappa^L}\Big)\Big)\Big) = 0. \tag{3.5}$$

Proof of Lemma 3.4: Assume the statement does not hold true, that is, for some $\varepsilon_1 > 0$ the left hand side of (3.5) is larger than ε_1. Then, one may find $\delta > 0$ arbitrarily small and a sequence $n_k = n_k(\delta, \varepsilon_1)$ such that (we write $L = L_{n_k}$)

$$\rho\Big(\mathcal{L}(\tilde{Z}_\delta^{(L)}(w)(n_k)), \Big(\mathbf{1}_{A_\delta(w)}(n_k) + \mathbf{1}_{A_\delta^c(w)}(n_k)\sqrt{\frac{r_L}{\delta\kappa^L}}\Big)\mathcal{N}(0,1)\Big) > \frac{\varepsilon_1}{2}. \tag{3.6}$$

Then, fixing $\delta_1 < \delta$, either one of the following occurs:

(a) There exists a further subsequence, still denoted by n_k, such that both (3.6) and $n_k \in A_\delta(w)$.

(b) There exists a further subsequence, still denoted by n_k, such that both (3.6) and $n_k \in A_\delta(w)^c \cap A_{\delta_1}(w)$.

(c) There exists a further subsequence, still denoted by n_k, such that both (3.6) and $n_k \in A_{\delta_1}(w)^c$.

Treating first case (a), one applies the Lindeberg–Feller theorem (see e.g. [4, p. 116, Theorem 4.5]). Indeed, one has on A_δ that $\tilde{Z}_\delta^{(L)}(w) = \tilde{Z}_0^{(L)}(w)$ and

$$\tilde{Z}_\delta^{(L)}(w) = \sum_{i=1}^{k_n} \frac{w\cdot(\tilde{X}_i^{(L)} - \tilde{\tau}_i^{(L)} v_L)}{\sqrt{k_n r_L}} =: \sum_{i=1}^{k_n} Y_{i,L}$$

and $\sum_{i=1}^{k_n} \overline{\mathbb{E}}^o Y_{i,L}^2 = 1$.

Next, we see that on A_δ, using Hölder's and then Chebycheff's inequalities in the first and second inequalities and $(\mathcal{A}5^\alpha)$ in the third,

$$\begin{aligned}
\sum_{i=1}^{k_n} \overline{\mathbb{E}}^o(Y_{i,L}^2 \cdot \mathbf{1}_{|Y_{i,L}|>\varepsilon}) &= \overline{\mathbb{E}}^o\Big(\frac{[w\cdot(\tilde{X}_1^{(L)} - \tilde{\tau}_1^{(L)} v_L)]^2}{r_L}\mathbf{1}_{\frac{|w\cdot(\tilde{X}_1^{(L)} - \tilde{\tau}_1^{(L)} v_L)|}{\sqrt{r_L}} \geq \varepsilon\sqrt{k_n}}\Big) \\
&\leq \frac{1}{(\varepsilon\sqrt{k_n})^{\alpha-2}}\overline{\mathbb{E}}^o\Big(\Big[\frac{w\cdot(\tilde{X}_1^{(L)} - \tilde{\tau}_1^{(L)} v_L)}{r_L^{1/2}}\Big]^\alpha\Big) \\
&\leq \frac{1}{(\varepsilon\sqrt{k_n})^{\alpha-2}(r_L)^{\alpha/2}}\overline{\mathbb{E}}^o(\tilde{\tau}_1^{(L)\alpha}) \\
&\leq \frac{M(L)}{\delta^{\alpha/2}\varepsilon^{\alpha-2}\kappa^{L\alpha/2}k_n^{(\alpha/2-1)}}
\end{aligned} \tag{3.7}$$

Using (1.24), one sees that the RHS in (3.7) converges to 0 with $n \to \infty$. This is enough in order to apply the Lindeberg–Feller theorem and conclude that for sequences $\{n_k\}$ in A_δ, $\tilde{Z}_\delta^{(L)}(w)$ converges in distribution to a standard Gaussian, contradicting (3.6).

Considering next case (b), the same argument as above proves that $\tilde{Z}_\delta^{(L)}(w)\sqrt{\delta\kappa^L/r_L}$ converges in distribution to a standard Gaussian. Hence, since the factor multiplying $\tilde{Z}_\delta^{(L)}(w)$ is uniformly bounded below by 1 on A_δ^c,

$$\rho(\mathcal{L}(\tilde{Z}_\delta^{(L)}(w)), \sqrt{\delta^{-1}\kappa^{-L}r_L}\mathcal{N}(0,1)) \underset{k\to\infty}{\longrightarrow} 0,$$

which again contradicts (3.6).

Finally, the proof of case (c) is a variance computation: Indeed, note that in that case,

$$\mathrm{Var}(\tilde{Z}_\delta^{(L)}(w)) = \frac{r_L}{\delta\kappa^L} \le \frac{\delta_1}{\delta}.$$

In particular, with δ_0 denoting the atom at 0, since

$$\sup\left\{\rho(\mu,\delta_0); \mu \text{ probability measure on } \mathbb{R}, \int x^2 d\mu < \frac{\delta_1}{\delta}\right\} \underset{\delta_1\to 0}{\longrightarrow} 0,$$

and

$$\sup\left\{\rho(\mathcal{N}(0,\sigma^2),\delta_0); \sigma^2 < \frac{\delta_1}{\delta}\right\} \underset{\delta_1\to 0}{\longrightarrow} 0,$$

we can choose δ_1 small enough (as a function of ε_1 and nothing else) such that the triangle inequality yields a contradiction to (3.6). ∎

The next step involves transferring results from $\tilde{S}_{k_n}^{(L)}$ to $X_{\tau_{k_n}^{(L)}}$. Toward this end, define the random variable

$$W_L = W_L(w,\delta) := \frac{w\cdot(X_{\tau_{k_n}^{(L)}} - \tau_{k_n}^{(L)} v_L)}{\kappa^{-L}\sqrt{k_n}\sqrt{r_L \vee \delta\kappa^L}}.$$

Lemma 3.8: *Assume k_n such that (1.24) and (1.25) hold. Then,*

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \rho\Big(\mathcal{L}(W_L), \mathcal{N}\Big(0, \Big(\mathbf{1}_{A_\delta(w)} + \mathbf{1}_{A_\delta^c(w)}\frac{r_L}{\delta\kappa^L}\Big)\Big)\Big) = 0. \tag{3.9}$$

Proof: Recall that

$$\tau_{k_n}^{(L)} = \kappa^{-L}\tilde{T}_{k_n}^{(L)} + \kappa^{-L}\sum_{i=1}^{k_n}\Delta_i^{(L)}(-\tilde{\tau}_i^{(L)} + Z_i^{(L)}), \tag{3.10}$$

and

$$X_{\tau_{k_n}^{(L)}} = \kappa^{-L}\tilde{S}_{k_n}^{(L)} + \sum_{i=L}^{k_n}\kappa^{-L}\Delta_i^{(L)}[-\tilde{X}_i^{(L)} + Y_i^{(L)}].$$

Since $|\tilde{X}_i^{(L)}| \leq \tilde{\tau}_i^{(L)}$ and $|Y_i^{(L)}| \leq Z_i^{(L)}$, Lemma 3.8 follows from Lemma 3.4 of this paper and [2] as soon as one shows that

$$\frac{\sum_{i=1}^{k_n} \Delta_i^{(L)} \tilde{\tau}_i^{(L)}}{\sqrt{k_n \kappa^L}} \xrightarrow{\text{Prob}} 0 \tag{3.11}$$

and

$$\sum_{i=1}^{k_n} \frac{\Delta_i^{(L)} Z_i^{(L)}}{\sqrt{k_n \kappa^L}} \xrightarrow{\text{Prob}} 0. \tag{3.12}$$

Let us prove (3.12), the proof of (3.11) being similar. We have

$$\begin{aligned}
\overline{\mathbb{E}}^o \sum_{i=1}^{k_n} \Delta_i^{(L)} Z_i^{(L)} \sqrt{\frac{\kappa^{-L}}{k_n}} &\leq \sqrt{\kappa^{-L} k_n} \sup_i \overline{\mathbb{E}}^o (\Delta_i^{(L)} Z_i^{(L)}) \\
&\leq \sqrt{k_n \kappa^{-L}} \overline{\mathbb{E}}^o (\Delta_1^{(L)})^{1/\alpha'} \sup_i \overline{\mathbb{E}}^o ((\Delta_i^{(L)} Z_i^{(L)})^\alpha)^{1/\alpha} \\
&\leq \sqrt{k_n \kappa^{-L}} M(L)^{1/\alpha} \phi'(L)^{1/\alpha'} \underset{L\to\infty}{\longrightarrow} 0,
\end{aligned}$$

by (1.25). ∎

We have completed the preliminaries to the

Proof of Theorem 3: The main issue is to control the error between X_n and $X_{\tau_{k_n}^{(L)}}$. In view of (3.10), note that $\overline{\mathbb{E}}^o \tilde{T}_{k_n}^{(L)} = k_n \overline{\mathbb{E}}^o \tilde{\tau}_1^{(L)} \geq k_n \frac{t_{av}}{2}$ and the argument in [2] (equation below (3.13)) shows that

$$\overline{\mathbb{E}}^o \frac{|\tau_{k_n}^{(L)} - \kappa^{-L} \tilde{T}_{k_n}^{(L)}|}{\kappa^{-L} \overline{\mathbb{E}}^o \tilde{T}_{k_n}^{(L)}} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.13}$$

Now, take $c_n = \kappa^{-2L} k_L [r_L \vee \delta \kappa^L]$. We start by showing that for all x,

$$\begin{aligned}
\lim_{\delta\to 0} \limsup_{n\to\infty} \Big| \overline{\mathbb{P}}^o \Big(\frac{X_n \cdot w - n v_L \cdot w}{\sqrt{c_n}} \leq x \Big) & \\
- \overline{\mathbb{P}}^o \Big(\mathcal{N}\Big(0, \Big(\mathbf{1}_{A_\delta(w)}(n) + \mathbf{1}_{A_\delta^c(w)}(n) \frac{r_L}{\delta \kappa^L}\Big)\Big) \leq x \Big) \Big| &= 0.
\end{aligned} \tag{3.14}$$

We have $\forall x, \delta_1, \delta_2$,

$$\begin{aligned}
\overline{\mathbb{P}}^o \Big(\frac{X_n \cdot w - n v_L \cdot w}{\sqrt{c_n}} \leq x \Big) &\leq \overline{\mathbb{P}}^o \Big(\Big| \frac{\tau_{k_n}^{(L)}}{n} - 1 \Big| > \delta_2 \Big) \\
&+ \overline{\mathbb{P}}^o \Big(\Big| \frac{\tau_{k_n}^{(L)}}{n} - 1 \Big| < \delta_2, \frac{|X_n \cdot w - X_{\tau_{k_n}^{(L)}} \cdot w - n v_L \cdot w + \tau_{k_n}^{(L)} v_L \cdot w|}{\sqrt{c_n}} > \delta_1, \Big) \\
&+ \overline{\mathbb{P}}^o \Big(\frac{X_{\tau_{k_n}^{(L)}} \cdot w - \tau_{k_n}^{(L)} v_L \cdot w}{\sqrt{c_n}} \leq x + \delta_1 \Big) := \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{3.15}$$

Using (3.13), (1.26), Lemma 3.1 and the estimates in Lemma 3.8, $I \to 0$ as $n \to \infty$. By Lemma 3.8, III $\to \Phi(\delta_1 + x)$. So, using a similar lower bound on the left most probability in (3.15), using the continuity of $\Phi(\cdot)$ and boundedness of the variance in (3.9), the claim (3.14) follows as soon as we prove that

$$\lim_{\delta_1 \to 0} \limsup_{\delta_2 \to 0} \limsup_{n \to \infty} \text{ II} = 0. \tag{3.16}$$

Recall that $\overline{\mathbb{E}}^o \tilde{\tau}_1^{(L)} \geq t_{av}/2$ for L large. Let

$$J = \Big\{ j : \Big| \frac{j\kappa^{-L}\overline{\mathbb{E}}^o \tilde{\tau}_1^{(L)}}{n} - 1 \Big| < 2\delta_2 \Big\}.$$

(We have $k_n \in J$ for large n.) Exactly as in the proof that $I \to 0$ as $n \to 0$, one has

$$p_0 \triangleq \overline{\mathbb{P}}^o \Big(\exists j \notin J : \Big| \frac{\tau_j^{(L)}}{n} - 1 \Big| < \delta_2 \Big) \underset{n\to\infty}{\longrightarrow} 0. \tag{3.17}$$

Write

$$\begin{aligned} \text{II} \leq\, & p_0 + \overline{\mathbb{P}}^o \Big(\max_{j \in J} \frac{1}{\sqrt{c_n}} |X_{\tau_j^{(L)}} \cdot w - X_{\tau_{k_n}^{(L)}} \cdot w - \tau_j^{(L)} v_L \cdot w + \tau_{k_n}^{(L)} v_L \cdot w| > \frac{\delta_1}{2} \Big) \\ & + \overline{\mathbb{P}}^o \Big(\max_{j \in J} \frac{1}{\sqrt{c_n}} |\tau_{j+1}^{(L)} - \tau_j^{(L)}| > \frac{\delta_1}{2} \Big) := p_0 + p_1 + p_2. \end{aligned} \tag{3.18}$$

Concerning p_2, note that for n large, using (1.26)

$$|J| \leq \frac{4\delta_2 n}{\kappa^{-L} \overline{\mathbb{E}}^o \tilde{\tau}_{k_n}^{(L)}} + 1 \leq c_1 \delta_2 k_n$$

for some constant c_1. Hence,

$$\begin{aligned} p_2 \leq\, & c_1 \delta_2 k_n \overline{\mathbb{P}}^o \Big(\kappa^{-L} \tilde{\tau}_1^{(L)} \geq \frac{\delta_1 \sqrt{c_n}}{4} \Big) \\ & + c_1 \delta_2 k_n \operatorname{ess\,sup} \overline{\mathbb{P}}^o \Big(\Delta_1 Z_1^{(L)} \kappa^{-L} \geq \frac{\delta_1 \sqrt{c_n}}{4} \Big| \mathcal{F}_0^L \Big) \\ := & p_{2,1} + p_{2,2}. \end{aligned}$$

But

$$\begin{aligned} p_{2,1} & \leq \frac{4^\alpha c_1 \delta_2 k_n}{\delta_1^\alpha c_n^{\frac{\alpha}{2}}} \kappa^{-\alpha L} \overline{\mathbb{E}}^o (\tilde{\tau}_1^{(L)})^\alpha \\ & \leq \frac{c_2 \delta_2 M(L)}{\delta_1^\alpha [r_L \vee \delta \kappa^L]^{\alpha/2}} \frac{k_n \kappa^{-\alpha L}}{k_n^{\frac{\alpha}{2}} \kappa^{-\alpha L}} \\ & \leq \frac{c_2 \delta_2}{\delta_1^\alpha \delta^{\alpha/2}} \frac{M(L)}{k_n^{\frac{\alpha}{2}-1} \kappa^{\alpha L/2}} \underset{L\to\infty}{\longrightarrow} 0 \end{aligned}$$

due to (1.24). Similarly, using [2, (3.10)]

$$p_{2,2} \le \frac{4^\alpha c_1 \delta_2 k_n}{\delta_1^\alpha c_n^{\frac{\alpha}{2}}} \kappa^{-\alpha L} M(L) \le c_3 \frac{\delta_2}{\delta_1^\alpha \kappa^{\alpha L/2}} k_n^{1-\frac{\alpha}{2}} M(L) \underset{L\to\infty}{\longrightarrow} 0$$

by (1.24). We conclude that

$$p_2 \underset{n\to\infty}{\longrightarrow} 0. \tag{3.19}$$

It thus only remains to treat p_1. As above, we can replace $X_{\tau_j^{(L)}}$ and $\tau_j^{(L)}$ by $\kappa^{-L}\tilde{S}_j^{(L)}$ and $\kappa^{-L}\tilde{T}_j^{(L)}$, on a set whose complement has probability smaller than

$$\overline{\mathbb{P}}^o\Big(\max_{j\in J} \frac{\kappa^{-L}}{\sqrt{c_n}} \Delta_j^{(L)} |\tilde{X}_j^{(L)}| > \frac{\delta_1}{8}\Big) + \overline{\mathbb{P}}^o\Big(\max_{j\in J} \frac{\kappa^{-L}}{\sqrt{c_n}} \Delta_j^{(L)} |Y_j^{(L)}| > \frac{\delta_1}{8}\Big),$$

which tends to zero. Now, since $\tilde{S}_j^{(L)} - \tilde{T}_j^{(L)} v_L = \sum_{i=1}^j (\tilde{X}_j^{(L)} - \tilde{\tau}_j^{(L)} v_L)$ is a series of i.i.d. random variables, one has, using Kolmogorov's inequality [4, p. 62], that

$$\overline{\mathbb{P}}^o\Big(\sup_{j\in J} \frac{\kappa^{-L}}{\sqrt{c_n}} |\tilde{S}_j^{(L)} \cdot w - \tilde{S}_{k_n}^{(L)} \cdot w - \tilde{T}_j^{(L)} v_L \cdot w + \tilde{T}_{k_n}^{(L)} v_L \cdot w| > \frac{\delta_1}{4}\Big) \le \frac{32|J|\kappa^{-2L}}{\delta_1^2 c_n} r_L \le c_4 \frac{\delta_2}{\delta_1^2}$$

since $r_L \le 2\overline{\mathbb{E}}^o(\tilde{\tau}_1^{(L)})^2$ is bounded independently of L by $(\mathcal{A}5^\alpha)$. Substituting the last inequality, (3.19) and (3.17) into (3.18) gives (3.16). This completes the proof of (3.14).

We end by proving that (3.14) implies Theorem 3 with

$$v(n) = v_L, \quad R_n(w) = \frac{\kappa^{-L}}{\overline{\mathbb{E}}^o \tilde{\tau}_1^{(L)}} \mathrm{Var}(w \cdot [\tilde{X}_1^{(L)} - \tilde{\tau}_1^{(L)} v_L]) \quad \text{where } L = L(n). \tag{3.20}$$

We argue by contradiction. If (1.27) does not hold for some w, take a subsequence n_k such that the left hand side is at least $\varepsilon > 0$. Moreover, going to a further subsequence if necessary, we can assume that $R_{n_k}(w)$ converges to a limit $R(w) \in [0,\infty]$. If $R(w)$ is positive and finite, this would contradict (3.14). If $R(w) = 0$, then

$$\kappa^{-L} \frac{\tilde{S}_{k_n} \cdot w - \overline{\mathbb{E}}^o \tilde{T}_{k_n}^{(L)} v_L \cdot w}{\sqrt{n}} \to 0 \quad \text{in } L^2,$$

then $(X_n \cdot w - n v_L \cdot w)/\sqrt{n}$ would tend to 0 in probability, yielding another contradiction. Now, if $R(w) = \infty$, the two terms in (1.27) of the Theorem tend to the same limit $0, 1/2, 1$ according to $x < 0$, $x = 0$, $x > 0$, yielding again a contradiction. This proves part (a) of (1.27).

Part (b) of the Theorem follows by setting

$$L = L(n) := \Big[\frac{\log n}{c + \log(1/\kappa)}\Big]_{|\ell|_1}, \quad k_n := \Big[\frac{n}{\kappa^{-L(n)}\overline{\mathbb{E}}^o \tilde{\tau}_1^{(L(n))}}\Big],$$

where $[z]_{|\ell|_1}$ denotes the largest element of $|\ell|_1\mathbb{N}^*$ not larger than z. Then, k_n satisfy trivially (1.26), but also (1.24), (1.25) and both sequences tend to ∞. Indeed, $\overline{\mathbb{E}}^o \tilde{\tau}_1^{(L(n))} \geq t_{av}/2$, and

$$\limsup_{n\to\infty} \overline{\mathbb{E}}^o \tilde{\tau}_1^{(L)} = \limsup_{n\to\infty} \frac{\overline{\mathbb{E}}^o \tilde{X}_1^{(L)} \cdot \ell}{v \cdot \ell}$$

(since the ratio of the two expectations tends to $v \cdot \ell$; see sentence before [2, Theorem 3.4]), where $\tilde{X}_1^{(L)} \cdot \ell$ has exponential tails by [2, Lemma 5.3].

Finally, part (c) of the Theorem is immediate by observing that if $L = L_0$ then $\Delta_i^{(L)} = 0$, and the conclusion follows from the standard CLT for the sum of i.i.d. random variables of finite variance. ∎

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